

ON CIRCUMFERENTIALLY MEAN p -VALENT FUNCTIONS

BY
JAMES A. JENKINS

1. In the paper [4] was given a geometric proof of the precise bound for the third coefficient of normalized regular univalent functions in the unit circle, a result first proved by Löwner with his parametric method. The present paper has its inception in the observation that the proof of [4] with suitable technical modifications applies also to the case of circumferentially mean 1-valent functions in the sense of Biernacki [1]. This leads in a natural manner to the precise bound for the third coefficient of normalized regular p -valent functions in the unit circle with a p -fold zero at the origin.

2. We begin by proving a symmetrization result for certain doubly-connected Riemann domains. Let us denote by $\mathfrak{B}$ the class of doubly-connected Riemann domains D covering the w -plane obtained by slitting a simply-connected Riemann surface, covering neither the origin nor the point at infinity, along a piece-wise analytic arc γ on the open surface. Then we can obtain a corresponding circularly symmetrized domain D^* in the following manner. Let the portion of the domain D covering the circle $|w| = R$ have total angular Lebesgue measure $l(R)$ (which may be $+\infty$). Let $\mathfrak{D}$ be the Riemann surface swept out by the open arcs $-l(R)/2 < \Phi < l(R)/2$ where R, Φ are polar coordinates in the w -plane. Let $R_1 \leq R \leq R_2$ be the range of values for which γ covers a point on $|w| = R$. Then D^* is obtained by slitting $\mathfrak{D}$ along the arc $R_1 \leq R \leq R_2, \Phi = 0$. The symmetrization result in question is then

THEOREM 1. *Let D be a Riemann domain in $\mathfrak{B}$ of module M and let D^* , the corresponding symmetrized domain, have module M^* . Then*

$$M \leq M^*.$$

The modules here are the usual ones, i.e. for the class of curves separating the boundary components [3]. For the proof we may suppose that the boundary component Γ of D other than γ is an analytic curve since the result follows in general in the limit from this case. Let now $u(w)$ be a function harmonic for $w \in D$ taking the continuous boundary values 1 on γ and 0 on Γ . Let $u = u(\xi, \eta)$ be a surface lying over the w -plane, $w = \xi + i\eta$, such that $u(\xi, \eta) = u(w)$ where w covers $\xi + i\eta$. This surface will in general have self-intersections, nevertheless we can apply to it the process of circular symmetrization with respect to the half-plane $\xi > 0, \eta = 0$ [6, p. 194]. In this way we obtain a surface $u = u^*(\xi, \eta)$ likewise lying over the w -plane and which again may have self-intersections. The function u^* has as natural domain of

definition the surface $\mathfrak{D}$ and the subset of $\mathfrak{D}$ on which $0 < u^* < 1$ is precisely D^* . The standard argument now shows that the Dirichlet integral of u over D is not less than the Dirichlet integral of u^* over D^* [6, pp. 194, 185] and the inequality $M \leq M^*$ follows by the Dirichlet principle and the familiar connection of the Dirichlet integral with the module [4, p. 512].

3. Let us denote by F_p , p a positive integer, the class of functions regular in the unit circle, $|z| < 1$, which have power series expansion about $z=0$,

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \dots$$

and which are circumferentially mean p -valent. By the latter we mean that if f maps $|z| < 1$ on a Riemann surface $\mathfrak{R}$ covering the w -plane the total angular measure of the open arcs on $\mathfrak{R}$ covering $|w| = r$, all $r > 0$, is at most $2\pi p$ [1].

In the paper [4] we constructed a continuous one-parameter family $f_t(z)$, $0 \leq t \leq 1$, of normalized regular univalent functions in the unit circle associated with a pair of values r_1, r_2 , $0 < r_i < 1$, $i=1, 2$, and having the following mapping properties. To each function $f_t(z)$ is associated a quadratic differential

$$d\zeta^2 = \pm \frac{(w-a)}{w^2(w-b)(w-c)} dw^2 \equiv Q(w)dw^2$$

where $b < 0$, $c > 0$ and $a < b$ or $c < a$ or for one particular value t_0 of t , a coincides with the point at infinity, the factor in the numerator of $Q(w)$ being replaced by unity. The sign is to be chosen so that $d\zeta^2$ is positive on the real axis near the origin. By the trajectories of the quadratic differential we mean the maximal open arcs or topological circles on which $Q(w)dw^2 > 0$. The maximal open arcs or topological circles on which $Q(w)dw^2 < 0$ are called orthogonal trajectories. There are three trajectories which have a limiting end point at a , T_1 running out along the real axis to infinity and T_2, T_3 symmetric in the real axis and tending to the origin. In the particular case where a coincides with the point at infinity, $t=t_0$, these degenerate to a single trajectory through the point at infinity. Now f_t maps $|z| < 1$ onto the w -plane slit along T_1 and along equiangular arcs on the closures of T_2, T_3 having an end point at $w=a$. In particular the functions $f_t(z)$ have real coefficients in their power series expansions about the origin. Further f_t maps $z=-r_1$ into $w=b$ and $z=r_2$ into $w=c$. Finally $f_0(z)=z(1-z)^{-2}$, $f_1(z)=z(1+z)^{-2}$ and the function $f_t(z)$ passes continuously from one to the other as t runs over $0 \leq t \leq 1$.

THEOREM 2. If $f \in F_1$ and $|f(-r_1)| = |f_t(-r_1)|$ then

$$|f(r_2)| \leq |f_t(r_2)|.$$

It should be observed that for $f \in F_1$

$$r_1/(1+r_1)^2 \leq |f(-r_1)| \leq r_1/(1-r_1)^2$$

as was proved by Hayman [2] (actually for a slightly larger class of functions) so that for every $f \in F_1$ there is a function f_t satisfying $|f(-r_1)| = |f_t(-r_1)|$, in fact precisely one. Hayman's result for the class F_1 can also be proved by a simple version of the present method.

For the proof of Theorem 2 we observe first that the orthogonal trajectories of $Q(w)dw^2$ in the neighborhood of $w=0$ are Jordan curves which approach circular form as they shrink down to the origin [7; 5]. Taking such an orthogonal trajectory L sufficiently small, together with the trajectories T_1, T_2, T_3 and the segment $b \leq \xi \leq c$ on the real axis it will bound two quadrangles R_1 and R_2 , each with a pair of opposite sides on L , R_1 having one further side along the segment σ_1 of the real axis from L to c described twice, R_2 having a corresponding side along the segment σ_2 of the real axis from b to L described twice. If $a < b$ the final side of R_1 is made up of arcs on T_1, T_2, T_3 , the final side of R_2 of arcs on T_2, T_3 ; if $c < a$ the situation is reversed.

Regard now the mappings $w=f_t(z)$, $w'=f(z)$ and the composite mapping $w'(w)$ from the image domain of $|z| < 1$ under f_t into the w' -plane. The mapping $w'(w)$ carries R_1 and R_2 into quadrangles R'_1 and R'_2 which are not necessarily schlicht and L into a closed curve L' . Let $R_i, i=1, 2$, have module M_i for the class of curves joining the pair of opposite sides on L . Then R'_i has the corresponding module also equal $M_i, i=1, 2$. We now choose L so small that $w'(w)$ is univalent on L , i.e. L' is a Jordan curve containing $w'=0$, that the image Riemann surface covers simply a circle N centered at $w'=0$, containing L' on the closed disc it bounds and touching L' from outside, and that N cuts off from $R'_i, i=1, 2$, just two domains, one at each end, leaving a quadrangle R''_i with a pair of opposite sides on N , the other sides lying along the corresponding sides of R'_i . The module M''_i of R''_i for the class of curves joining the pair of opposite sides lying on N satisfies $M''_i \geq M_i, i=1, 2$.

Reflecting $R'_i, i=1, 2$, in the circle N and joining it with its image across its boundary arcs on N we obtain a doubly-connected domain D_i in $\mathfrak{B}$. The boundary component of D_i arising from the side of R'_i corresponding to σ_i and its reflection in N plays the role of γ of §2. The domain D_i has module $M''_i/2, i=1, 2$, for the class of curves separating its boundary components. Applying to D_i the symmetrization process of §2 we obtain a domain D_i^* of not smaller module which in the present situation is schlicht. The circle N divides D_i^* into two quadrangles each with a pair of opposite sides on N and of equal module for the classes of curves joining these sides. Let the quadrangle arising from D_1^* and exterior to N be denoted by R_1^* . Let the quadrangle obtained by rotating through 180° the quadrangle arising from D_2^* and exterior to N be denoted by R_2^* . The quadrangle R_1^* has in addition to its sides on N a side along a segment of the positive real axis from N to a point of modulus $\geq |f(r_2)|$ described twice. The quadrangle R_2^* has in addition to its sides on N a side along a segment of the negative real axis from N to a point of modulus $\geq |f(-r_1)|$ described twice. The quadrangle R_i^* ,

$i = 1, 2$, has module $M_i^* \geq M_i'' \geq M_i$ for the class of curves joining the pair of opposite sides on N . The two quadrangles R_1^* and R_2^* do not overlap since f is in F_1 .

Now the assumption $|f(r_2)| > |f_i(r_2)|$ leads to a contradiction by the same argument as in [4, pp. 519, 520]. This completes the proof of Theorem 2.

THEOREM 3. *If $f(z) \in F_1$ and $0 < r_1 \leq r_2 < 1$, then*

$$|f(-r_1)| + |f(r_2)| \leq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2,$$

equality being attained for the function $z(1 - z)^{-2}$.

The proof is almost the same as for Theorem 4 in [4] which the present theorem extends. Because of its brevity it is repeated here. By Theorem 2, the maximum of $|f(-r_1)| + |f(r_2)|$ for r_1, r_2 fixed and f varying in the family F_1 is attained for a univalent function with real coefficients. For such

$$|f(-r_1)| + |f(r_2)| = r_2 + a_2 r_2^2 + \cdots + r_1 - a_2 r_1^2 + \cdots.$$

For functions with real coefficients we have $|a_n| \leq n$ and since $r_1 \leq r_2$

$$\begin{aligned} |f(-r_1)| + |f(r_2)| &\leq r_2 + 2r_2^2 + 3r_2^3 + \cdots + r_1 - 2r_1^2 + 3r_1^3 - \cdots \\ &\leq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2. \end{aligned}$$

The statement concerning equality is evident.

COROLLARY 1. *If $f(z) \in F_1$, $0 < r_1 \leq r_2 < 1$, θ real, then*

$$|f(-r_1 e^{i\theta})| + |f(r_2 e^{i\theta})| \leq r_2/(1 - r_2)^2 + r_1/(1 + r_1)^2,$$

equality being attained for the function $z(1 - ze^{-i\theta})^{-2}$.

COROLLARY 2. *If $f(z) \in F_1$, $0 < r < 1$, then, for $|z_1| = r$*

$$|f(-z_1)| + |f(z_1)| \leq 2r(1 + r^2)/(1 - r^2)^2,$$

equality being attained for the function $z(1 - ze^{-i\theta})^{-2}$ with $\theta = \arg z_1$.

COROLLARY 3. *If $f(z) \in F_1$, then $|a_3| \leq 3$.*

Indeed, if $z = re^{i\theta}$, then

$$|f(-z) + f(z)| \leq |f(-z)| + |f(z)| \leq 2r(1 + r^2)/(1 - r^2)^2.$$

Thus

$$r(1 + \Re(a_3 z^2) + O(r^4)) \leq r(1 + 3r^2 + O(r^4))$$

or

$$\Re(a_3 e^{2i\theta}) \leq 3.$$

Proper choice of θ gives $|a_3| \leq 3$.

It should be observed that the corresponding result does not hold for the class of areally mean 1-valent functions [8] as is shown by an unpublished result due to A. C. Schaeffer and D. C. Spencer.

THEOREM 4. *If $f(z) \in F_p$, then*

$$(1) \quad \left| \frac{a_{p+2}}{p} + \frac{1}{2p} \left(\frac{1}{p} - 1 \right) a_{p+1}^2 \right| \leq 3.$$

This bound is sharp, equality being attained for the functions $z^p(1 - ze^{i\theta})^{-2p}$, θ real.

Indeed, with a suitable choice of determination, the function $(f(z))^{1/p}$ belongs to F_1 and has power series expansion about the origin

$$(f(z))^{1/p} = z + \frac{a_{p+1}}{p} z^2 + \left(\frac{a_{p+2}}{p} + \frac{1}{2p} \left(\frac{1}{p} - 1 \right) a_{p+1}^2 \right) z^3 + \dots$$

The inequality (1) then follows immediately by Corollary 3. The statement concerning equality is evident.

COROLLARY 4. *If $f(z) \in F_p$, then*

$$|a_{p+2}| \leq 2p^2 + p.$$

This bound is sharp, equality being attained for the functions $z^p(1 - ze^{i\theta})^{-2p}$, θ real.

Indeed $|a_{p+1}| \leq 2p$ as was proved by Spencer [8] for the larger class of areally mean p -valent functions. Thus by inequality (1)

$$\frac{|a_{p+2}|}{p} \leq 3 + \frac{1}{2p} \left(1 - \frac{1}{p} \right) 4p^2$$

or

$$|a_{p+2}| \leq 2p^2 + p.$$

The statement concerning equality is evident.

In case $p > 1$ equality can be attained only for the extremal functions indicated, since it is only for these functions that equality can occur in Spencer's result.

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UNIVERSITY OF NOTRE DAME,
NOTRE DAME, IND.